SPECIAL FEATURES OF THE DYNAMICS OF A SPHERICAL GAS BUBBLE IN A LIQUID

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In the majority of problems connected with the dynamics of a liquid containing gas bubbles, the nature of the pulsations of individual bubbles has great significance. In many cases the additional pressure field established by these pulsations is the deciding factor in determining the overall state of the medium containing the bubbles. In what follows we consider some properties of the pulsations of a spherical gas bubble in compressible and incompressible liquids.

1. The motion of the wall of a spherical bubble in an incompressible fluid is governed by the following equation (viscosity is not taken into account):

$$RR^{\cdot\cdot} + {}^3/_2 R^{\cdot 2} = (P(R) - P(t)) / \rho. \tag{1.1}$$

Here P(R) is the pressure inside the bubble, P(t) is the applied pressure, ρ is the density of the liquid, and R is the radius of the bubble; a dot indicates the total derivative with respect to time. For P(t) = const, and on the condition that the bubble be compressed adiabatically, it is easy from Eq. (1.1) to obtain

$$(R_0/R_*)^{3\gamma-3} = 1 + A(\gamma - 1) \quad (A = P/P_0) \tag{1.2}$$

(where γ is the adiabatic exponent, R_* is the minimum radius of the cavity, R_0 is the initial radius, P_0 is the initial pressure in the bubble), and to determine the compression time of the cavity

$$t = 0.915 R_0 \sqrt{\rho/p}\ . \tag{1.3}$$

However, we usually have to deal with pressure essentially as a function of time. In this case neither the time nor degree of bubble compression can be determined directly from (1.1).

In [1] the results are presented from numerical solution of Eq. (1.1) in dimensionless form

$$y\frac{d^2y}{dz^2} + \frac{3}{2}\left(\frac{dy}{dz}\right)^2 = \mu\left(\frac{1}{y^{3\gamma}} - Ae^{-z}\right)$$

$$\left(y = \frac{R}{R_0},\ z = \frac{t}{\tau},\ \mu = \left(\frac{\tau}{R_0\sqrt{\rho/p_0}}\right)^2\right).$$

Here μ is a dimensionless parameter which determines the ratio of the time constant for pressure decrease to the characteristic time of bubble compression by the constant pressure P_0. The calculations are given for the case of waves with an exponential profile with A = 10, 100, and 1000 while μ varies from 0.01 to 1000. The analysis shows that the pulsation of a bubble under pressure with various values of τ obeys a definite law

$$(t_2^* / \tau_2) = (1.2 - k\sqrt{\mu_2/\mu_1})^k (\mu_1/\mu_2)^{1/2} (t_1^*/\tau_1). \tag{1.4}$$

Here t_2^* is the required time of bubble compression for waves with a time constant of τ_2 when t_1^* is known for a wave with equal amplitude but a time constant of τ_1 (the pressure in the wave front enters into Eq. (1.4) by means of t_1^*). Superscript k is determined from the condition

$$10^k = \left(\frac{\mu_1}{\mu_2}\right)^{1/2}. \tag{1.5}$$

We see from (1.4) that the connection between the times of bubble compression by pressure waves with various values of τ is determined to within a constant coefficient by the square root of the ratio of the dimensionless parameters μ which are characteristic of these waves. If we know the time of bubble compression by a wave with constant pressure behind its front, expression (1.4) enables us to calculate, for example, the time of bubble compression by a wave of the same amplitude but with very small τ, while μ_1 is chosen so as to satisfy (1.3).

Another characteristic of a pulsating bubble is the minimum compression radius. The connection between the minimum radius, pressure amplitude, and characteristics of a constant-pressure wave can, by analogy with expression (1.2), be written in the form

$$\left(\frac{R_0}{R_*}\right)^{3\gamma-3} = 1 + \frac{\mu A^2(\gamma - 1)}{1 + \mu A}. \tag{1.6}$$

Clearly, this expression passes to (1.2) as $\mu \to \infty$, i.e., for waves with constant pressure behind the wave front.

Values calculated from Eqs. (1.4) and (1.6) for various A and μ (R_0 = 1 cm) are given in Tables 1 and 2, where they are compared with data from [1] (in the tables the results of machine calculations are denoted by a subscripted 1 and those made from relations (1.4) and (1.6) by a subscripted 2). Expressions (1.4) and (1.6) are quite suitable for making approximate estimates of the basic characteristics of bubble pulsation in an incompressible fluid under a pressure which varies greatly with time.

2. Limitation of the discussion of bubble pulsations to the case of an incompressible fluid leads to a considerable discrepancy between the observed and calculated pulsation characteristics when we consider cases in which the cavity walls reach a velocity on the order of the speed of sound. This occurs, for example, in problems of cavitation and its accompanying phenomena. When the theory of the collapse of empty bubbles was first formulated it was concluded that compressibility had to be taken into account, in view of the enormous velocities and pressures encountered as a result of collapse. The same thing is observed in the compression of a gas-filled cavity under very high pressure. We consider the spherically symmetrical problem of the pulsation of a gas bubble in a compressible nonviscous fluid. The particle velocity $U^{(1)}$ is replaced by the velocity-potential gradient φ, and the system of equations is written as

$$\frac{\partial}{\partial t}(-\nabla\varphi) + (\mathbf{U}\cdot\nabla)\mathbf{U} = -\frac{\nabla P}{\rho},$$

$$\nabla\cdot\mathbf{U} = -\frac{1}{\rho}\frac{d\rho}{dt}. \tag{2.1}$$

Here P is the pressure and ρ is the liquid density. Integration of (2.1) yields

$$-\frac{\partial\varphi}{\partial t} + \frac{v^2}{2} = -\int\limits_{P_\infty}^{P}\frac{dP}{\rho} = -h, \tag{2.2}$$

if h is the enthalpy difference between point r and infinity. It is assumed that P_∞ is constant at infinity, that the velocity and the velocity potential vanish at infinity, and that ρ is a function of pressure only. When

$$\varphi = \frac{1}{r}f\left(t - \frac{r}{c}\right), \tag{2.3}$$

we have

$$r(h + {}^1/_2 U^2) = f'(t - r/C). \tag{2.4}$$

Equations (2.3) and (2.4) show that $r\varphi$ and $r(h + (U^2/2))$ are propagated with velocity C (local speed of sound), in the acoustic approximation. Kirkwood [2] made the assumption that $r(h + (U^2/2))$ is propagated with a velocity C + U, on the basis that liquid velocities can attain values on the order of the speed of sound. In view of the

Table 1

A	μ	γ	τ, sec	$(R^*/R_0)_1$	$(R^*/R_0)_2$
10	∞ (10)	1.4	0.00316	0.262	0.262
10	1	1.4	0.001	0.307	0.280
10	0.1	1.4	0.000316	0.404	0.400
10	0.01	1.4	0.0001	0.738	0.772
100	∞ (10)	1.4	0.00316	0.046	0.046
100	1	1.4	0.001	0.048	0.0457
100	0.1	1.4	0.000316	0.053	0.0490
100	0.01	1.4	0.0001	0.074	0.0793
10	∞ (10)	1.33	0.00316	0.247	0.235
10	1	1.33	0.001	0.277	0.250
10	0.1	1.33	0.000316	—	0.377
10	0.01	1.33	0.001	0.732	0.768
100	∞ (10)	1.33	0.00316	0.030	0.0294
100	1	1.33	0.001	0.031	0.0306
100	0.1	1.33	0.000316	0.035	0.0323
100	0.01	1.33	0.0001	0.052	0.0570
1000	∞ (1)	1.33	0.001	0.00304	0.00302
1000	0.1	1.33	0.000316	0.00317	0.00302
1000	0.01	1.33	0.0000	0.00358	0.00333
10	∞ (10)	1.67	0.00316	0.382	0.361
10	1	1.67	0.001	0.413	0.376
10	0.1	1.67	0.000316	0.498	0.48
10	0.01	1.67	0.0001	0.764	0.78
100	∞ (10)	1.67	0.00316	0.123	0.121
100	1	1.67	0.001	0.126	0.122
100	0.1	1.67	0.000316	0.136	0.127
100	0.01	1.67	0.0001	0.167	0.170
1000	∞ (1)	1.67	0.001	0.0386	0.0387
1000	0.1	1.67	0.000316	0.040	0.0390

Table 2

A	μ	γ	τ, sec	t_1^*/τ_1	t_2^*/τ_2
10	1000	1.4	0.0316	0.01	0.01
10	100	1.4	0.0100	0.0325	0.0323
10	10	1.4	0.00316	0.105	0.110
10	1	1.4	0.001	0.35	0.37
10	0.1	1.4	0.000316	1.27	1.38
10	0.01	1.4	0.0001	5.11	5.00
100	10	1.4	0.00316	0.0294	0.0294
100	1	1.4	0.001	0.0939	0.0950
100	0.1	1.4	0.000316	0.308	0.323
100	0.01	1.4	0.0001	1.096	1.130
10	100	1.33	0.01	0.0325	0.0323
10	10	1.33	0.00316	0.105	0.110
10	1	1.33	0.001	0.350	0.373
10	0.01	1.33	0.0001	5.11	5.00
100	10	1.33	0.00316	0.0294	0.0294
100	1	1.33	0.001	0.0939	0.0950
100	0.1	1.33	0.000316	0.307	0.323
100	0.01	1.33	0.0001	1.094	1.130
1000	1	1.33	0.001	0.029	0.029
1000	0.1	1.33	0.000316	0.093	0.094
1000	0.01	1.33	0.0001	0.304	0.319
10	10	1.67	0.00316	0.105	0.110
10	1	1.67	0.001	0.35	0.373
10	0.1	1.67	0.000316	1.19	1.380
10	0.01	1.67	0.0001	5.11	5.00
100	10	1.67	0.00316	0.0295	0.0295
100	1	1.67	0.001	0.0944	0.095
100	0.1	1.67	0.000316	0.309	0.323
100	0.01	1.67	0.0001	1.100	1.130
1000	1	1.67	0.001	0.029	0.029
1000	0.1	1.67	0.000316	0.093	0.094

assumptions made [3, 4] we write

$$\frac{\partial}{\partial t}\left[r\left(h+\frac{U^2}{2}\right)\right]=-(C+U)\frac{\partial}{\partial r}\left[r\left(h+\frac{U^2}{2}\right)\right]. \tag{2.5}$$

Expanding Eq. (2.5) with

$$\frac{dU}{dt}=-\frac{\partial h}{\partial r},\ \frac{\partial U}{\partial r}+\frac{2U}{r}=-\frac{1}{C^2}\frac{dh}{dt}$$
$$\left(\frac{d}{dt}=\frac{\partial}{\partial t}+U\frac{\partial}{\partial r}\right), \tag{2.6}$$

we obtain the law governing the motion of the wall of a gas bubble

$$RR^{\cdot\cdot}\left(1-\frac{R^{\cdot}}{C}\right)+\frac{3}{2}R^{\cdot 2}\left(1-\frac{1}{3}\frac{R^{\cdot}}{C}\right)=$$
$$=H\left(1+\frac{R^{\cdot}}{C}\right)+\frac{RH^{\cdot}}{C}\left(1-\frac{R^{\cdot}}{C}\right). \tag{2.7}$$

We must, however, explain to what degree the equation obtained (2.7) corresponds to the exact equations of flow (2.1). Clearly, to define most fully the applicability of Kirkwood's approximation we must consider the case of compression of an empty cavity. This enables us to investigate the behavior of the function obtained, and to do so within a wide range of velocities of the cavity wall (from 0 to ∞).

The numerical integration of Eqs. (2.1) performed by Hunter [5] for a spherically symmetrical empty cavity in water revealed [the existence of] large flow velocities close to the point of collapse. It was found that in this case the radius of the cavity was proportional to $(-t)^n$ ($t = 0$ is the moment of collapse). The flow in the neighborhood of the point of collapse is described by a self-similar solution from which the quantity n is determined. In [5, 6] n was found equal to 0.5552. Writing Eq. (2.7) for the case of an empty cavity (i.e., setting C = const and H = const), we obtain

$$RR^{\cdot\cdot}\left(1-\frac{R^{\cdot}}{C}\right)+\frac{3}{2}R^{\cdot 2}\left(1-\frac{1}{3}\frac{R^{\cdot}}{C}\right)=H\left(1+\frac{R^{\cdot}}{C}\right). \tag{2.8}$$

Hence

$$\left(\frac{R_0}{R}\right)^3=\left(1-\frac{1}{3}\frac{R^{\cdot}}{C}\right)^4\left[1+\frac{3}{2}\frac{C^2}{-H}\left(\frac{R^{\cdot}}{C}\right)^2\right]. \tag{2.9}$$

Substituting Hunter's solution [5] in the form $R \sim At^n$ into Eq. (2.9), we obtain the value of n as $t \to 0$ without difficulty. It is equal to 0.666. For the case of an incompressible fluid n = 0.4, i.e., in the neighborhood of the point of collapse the behavior of the cavity wall according to Kirkwood's approximation departs from Hunter's result to the same extent as in the incompressible case. Repeating these considerations for the acoustic case, we have

$$RR^{\cdot\cdot}\left(1+\frac{2R^{\cdot}}{C}\right)+\frac{3}{2}R^{\cdot 2}\left(1-\frac{4}{3}\frac{R^{\cdot}}{C}\right)=$$
$$=H+\frac{RH^{\cdot}}{C}\left(1-\frac{R^{\cdot}}{C}+\frac{R^{\cdot 2}}{C^2}\right), \tag{2.10}$$

which yields a value of n = 0.5 in the case of an empty cavity, i.e., the exact solution lies between the acoustic case and Kirkwood's

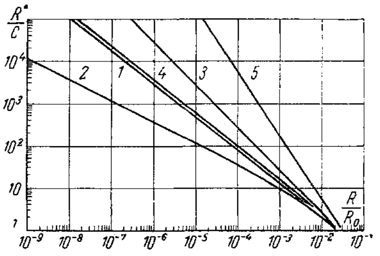

Fig. 1

calculations. It should be noted that Eq. (2.10) is just Hering's [2] equation, although the latter was obtained in a way different from that described above. From the values of n obtained we can conclude that the propagation velocity of the quantity $r(h + U^2/2)$ lies between C and C + U. We shall assume that propagation occurs with a velocity $C + \alpha U$, where α = const. In this case we have

$$\left(\frac{R_0}{R}\right)^3=\left[1+\frac{3}{2}\frac{C^2}{-H}\left(\frac{R^{\cdot}}{C}\right)^2\right]\left[1+\left(\alpha-\frac{4}{3}\right)\frac{R^{\cdot}}{C}\right]^{-{}^4/_3/(\alpha-{}^4/_3)}. \tag{2.11}$$

Substituting $R = At^{0.555}$ into this equation we obtain the value of α in the neighborhood of the point of collapse (for an infinite wall velocity) without difficulty; it is equal to 0.57. Analysis of the behavior of Eq. (2.11) for various values of $R^{\cdot}/C$ shows that to some degree of approximation (at each moment the value of α corresponding to Hunter's curve was found) α is a monotonically decreasing function of $R^{\cdot}/C$, which varies between 1 and 0.57 as $R \to 0$. However, this does not exclude the possibility of describing the process of collapse approximately by means of some constant value of α. It should be noted that Eq. (2.11) is quite convenient, since for various values of α it can pass to Kirkwood's equation (α = 1), to the equation for

Table 3

$-R^{\cdot}/C$	$(R/R_0)_1$	$(R/R_0)_2$	$(R/R_0)_3$	$(R/R_0)_4$	$(R/R_0)_5$
1.46	1.48 10^{-2}	1.41 10^{-2}	1.68 10^{-3}	1.54 10^{-2}	2.40 10^{-2}
2.05	1.00 10^{-2}	9.56 10^{-3}	1.24 10^{-2}	1.10 10^{-2}	1.92 10^{-2}
2.50	7.84 10^{-3}	7.48 10^{-3}	1.03 10^{-2}	8.92 10^{-3}	1.68 10^{-2}
2.93	6.42 10^{-3}	6.08 10^{-3}	8.90 10^{-3}	7.55 10^{-3}	1.51 10^{-2}
3.56	5.00 10^{-3}	4.66 10^{-3}	7.40 10^{-3}	6.10 10^{-3}	1.33 10^{-2}
4.00	4.27 10^{-3}	3.97 10^{-3}	6.62 10^{-3}	5.37 10^{-3}	1.23 10^{-2}
4.62	3.56 10^{-3}	3.22 10^{-3}	5.80 10^{-3}	4.56 10^{-3}	1.12 10^{-2}
5.50	2.85 10^{-3}	2.49 10^{-3}	4.90 10^{-3}	3.73 10^{-3}	1.00 10^{-2}
6.88	2.14 10^{-3}	1.75 10^{-3}	3.94 10^{-3}	2.87 10^{-3}	8.55 10^{-3}
9.50	1.43 10^{-3}	1.03 10^{-3}	2.96 10^{-3}	1.96 10^{-3}	6.90 10^{-3}
11.30	1.14 10^{-3}	7.66 10^{-4}	2.44 10^{-3}	1.59 10^{-3}	6.13 10^{-3}
19.50	5.70 10^{-4}	2.91 10^{-4}	1.43 10^{-3}	8.20 10^{-4}	4.26 10^{-3}
50.00	1.72 10^{-4}	4.90 10^{-5}	5.62 10^{-4}	2.56 10^{-4}	2.28 10^{-3}
80.00	9.42 10^{-5}	2.10 10^{-5}	3.51 10^{-4}	1.41 10^{-4}	1.67 10^{-3}
100.00	7.16 10^{-5}	1.34 10^{-5}	2.81 10^{-4}	1.07 10^{-4}	1.44 10^{-3}
200.00	2.97 10^{-5}	3.33 10^{-6}	1.41 10^{-4}	4.38 10^{-5}	9.08 10^{-4}
300.00	1.78 10^{-5}	1.49 10^{-6}	9.40 10^{-5}	2.62 10^{-5}	6.92 10^{-4}
400.00	1.24 10^{-5}	8.40 10^{-7}	7.02 10^{-5}	1.82 10^{-5}	5.70 10^{-4}
500.00	9.27 10^{-6}	5.33 10^{-7}	5.60 10^{-5}	1.36 10^{-5}	4.90 10^{-4}
600.00	7.37 10^{-6}	3.72 10^{-7}	4.70 10^{-5}	1.07 10^{-5}	4.35 10^{-4}
700.00	6.06 10^{-6}	2.72 10^{-7}	4.01 10^{-5}	8.93 10^{-6}	3.94 10^{-4}
800.00	5.11 10^{-6}	2.09 10^{-7}	3.52 10^{-5}	7.59 10^{-6}	3.58 10^{-4}
900.00	4.38 10^{-6}	1.65 10^{-7}	3.13 10^{-5}	6.47 10^{-6}	3.33 10^{-4}
1000.0	3.85 10^{-6}	1.34 10^{-7}	2.82 10^{-5}	5.65 10^{-6}	3.10 10^{-4}
10000.0	2.07 10^{-7}	1.34 10^{-9}	2.82 10^{-6}	3.00 10^{-7}	6.60 10^{-5}
100000.0	1.11 10^{-8}	1.34 10^{-11}	2.82 10^{-7}	1.62 10^{-8}	1.44 10^{-5}

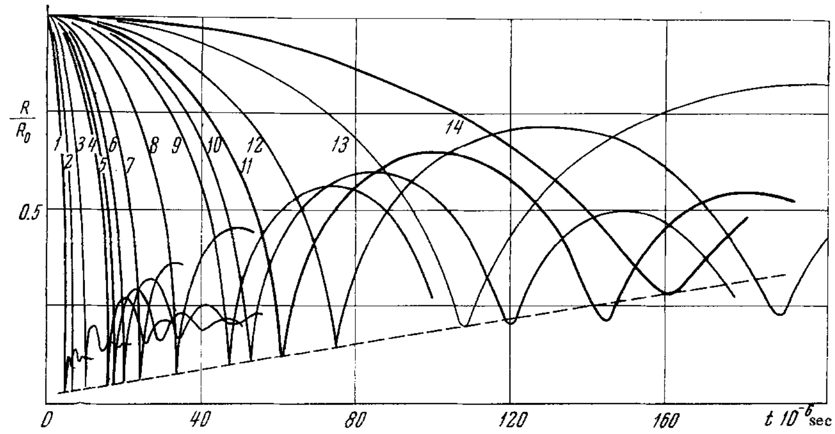

Fig. 2

the acoustic approximation ($\alpha = 0$), or to any other intermediate equation.

Table 3 gives results for the following cases:

(1) Hunter's numerical integration of the equations of the flow;

(2) calculations on Kirkwood's assumption from the equation

$$\left(\frac{R_0}{R}\right)^3 = \left[1 + \frac{3}{2}\frac{C^2}{-H}\left(\frac{R^{\cdot}}{C}\right)^2\right]\left[1 - \frac{1}{3}\frac{R^{\cdot}}{C}\right]^4; \tag{2.12}$$

(3) calculations of the acoustic variant

$$\left(\frac{R_0}{R}\right)^3 = \left[1 + \frac{3}{2}\frac{C^2}{-H}\left(\frac{R^{\cdot}}{C}\right)^2\right]\left[1 - \frac{4}{3}\frac{R^{\cdot}}{C}\right]; \tag{2.13}$$

(4) calculations on the assumption that the propagation velocity is C + 0.6 U

$$\left(\frac{R_0}{R}\right)^3 = \left[1 + \frac{3}{2}\frac{C^2}{-H}\left(\frac{R^{\cdot}}{C}\right)^2\right]\left[1 - 0.73\frac{R^{\cdot}}{C}\right]^{1.82}; \tag{2.14}$$

(5) calculations for an incompressible fluid

$$\left(\frac{R_0}{R}\right)^3 = \left[1 + \frac{3}{2}\frac{C^2}{-H}\left(\frac{R^{\cdot}}{C}\right)^2\right]. \tag{2.15}$$

The quantity $R^{\cdot}/C$ as a function of R/R_0 is shown graphically in Fig. 1, as developed from the data of the table. We note that Kirkwood's assumption is valid for $R^{\cdot}/C$ on the order of unity or less [7], but that it is not satisfied for high velocities. This is quite reasonable when we remember that the velocity C + U is valid for the two-dimensional case. The acoustic variant also exhibits a fairly marked departure from Hunter's curve, and in the region of $R^{\cdot}/C$ from 1 to 10 is much more at variance with the exact data than Kirkwood's curve. The arbitrary coefficient α introduced into the propagation velocity enables us to make some estimates about the behavior of the quantity C + U as the velocity of the bubble wall increases.

The results obtained were taken into account when we calculated the pulsation of a spherical bubble, 1 cm in diameter, under a suddenly applied constant pressure with an amplitude which varied in the range from 10 to 18 000 atm. An air bubble with an initial pressure of 1 atm was considered. Equation (2.7) was calculated on an electronic computer. The equation of state for water was taken as given in [2]. Results of the calculations are given in Fig. 2. The numbers correspond to the pressure amplitudes in atm — 18 000, 9000, 3000, 1000, 800, 600, 400, 200, 100, 80, 60, 40, 20, and 10. The dashed straight line joins all first pulsation minima. It can easily be established from the graph that R^*/R_0 is directly proportional to the compression time of the cavity:

$$R^* / R_0 = At_{t_*} + 0.025\,. \tag{2.16}$$

The compression time t_* is fairly accurately specified by expression (1.3), while $A = (5/3)10^3\ \mathrm{sec}^{-1}$ is easily found from the graph.

Some experiments were performed in a hydrodynamic shock tube to determine air-bubble pulsations at pressures of several hundred atmospheres with a weakly varying pressure behind the [shock] front. A description of both the experimental procedure and the apparatus is presented in [8], where the characteristic development of bubble pulsation with time is also given. The data for the degree and time of compression correspond to the calculations.

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